Continuation of this question:

$\boxed{\text{Example}}$ Let H be the subgroup of $S_7 = G$ generated by $(1,2,3,4)$ and $(5,6)$.

① Find $C_G(H)$

② Find $N_G(H)$

---

① Since $(1,2,3,4)$ & $(5,6)$ commute,

$$H = \{\underset{\tau^4}{e}, \underset{\tau}{(1,2,3,4)}, (1,3)(2,4), (1,4,3,2), \overset{\alpha}{(5,6)}, (1,2,3,4)(5,6), (1,3)(2,4)(5,6), (1,4,3,2)(5,6)\}$$

$= \{\tau^j \alpha^k : \tau^4 = e = \alpha^2, \alpha\tau = \tau\alpha\}$; $\{e, \tau, \tau^2, \tau^3, \alpha, \tau\alpha, \tau^2\alpha, \tau^3\alpha\}$ abelian $= \langle \alpha, \tau : \tau^4 = e, \alpha^2 = e, \alpha\tau = \tau\alpha \rangle$; $\alpha\tau = \tau\alpha$, $\alpha^2 = \tau^4 = e$

$H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$

If $\sigma \in C_G(H) \Leftrightarrow \sigma h \sigma^{-1} = h \quad \forall h \in H.$

$\Leftrightarrow \sigma\tau\sigma^{-1} = \tau$
$\quad\;\; \sigma\alpha\sigma^{-1} = \alpha.$

$\sigma\tau\sigma^{-1} = \tau$
$\Rightarrow \sigma\tau^2\sigma^{-1} = \sigma\tau\sigma^{-1}\sigma\tau\sigma^{-1}$
$= \tau^2$

$\Leftrightarrow \sigma(1,2,3,4)\sigma^{-1} = (1,2,3,4) = (2,3,4,1) \ldots$
$\quad\;\; \sigma(5,6)\sigma^{-1} = (5,6)$

$\Leftrightarrow (\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (1,2,3,4)$
$\quad\;\; (\sigma(5), \sigma(6)) = (5,6)$

$$\Leftrightarrow \begin{array}{lclcl} \sigma(1) = 1 & & 2 & & 3 & & 4 \\ \sigma(2) = 2 & \text{or} & 3 & \text{or} & 4 & \text{or} & 1 \\ \sigma(3) = 3 & & 4 & & 1 & & 2 \\ \sigma(4) = 4 & & 1 & & 2 & & 3 \end{array}$$

$\tau.$

and $\quad \begin{array}{lcl} \sigma(5) = 5 & \text{or} & 6 \\ \sigma(6) = 6 & & 5 \end{array}$

(and $\quad \sigma(7) = 7$)

$\Leftrightarrow \sigma \in$ Group generated by $(1,2,3,4)$ & $(5,6)$

So $C_G(H) = H.$

[Note: If we were considering H as being a subgroup of S_8, then $C_G(H)$ contains H & $(7,8)$.
$\Rightarrow C_G(H) = H \cup (7,8)H$]

Since $C_G(H) = H$, H is abelian
(In fact $H \cong \mathbb{Z}_4 \times \mathbb{Z}_2$).

The Normalizer $N_G(H)$ is

$$\{\sigma \in S_7 : \sigma H \sigma^{-1} = H\}$$

Each element of H is ($\tau = (1,2,3,4)$
$\alpha = (5,6)$)

$$\tau^j \alpha^k.$$

If $\sigma\tau\sigma^{-1} \in H$ and $\sigma\alpha\sigma^{-1} \in H$
Then $\sigma h \sigma^{-1} \in H \ \forall h \in H$.

(Proof: $\sigma \tau^j \alpha^k \sigma^{-1} = (\sigma\tau\sigma^{-1})^j (\sigma\alpha\alpha^{-1})^k$

So if the generators satisfy $\sigma\tau\sigma^{-1} \in H$,
$\sigma\alpha\sigma^{-1} \in H$,
then that suffices.)

$\sigma\tau\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ ← must be a 4-cycle in H

$\sigma\alpha\sigma^{-1} = (\sigma(5), \sigma(6))$ ← must be a 2-cycle in H.

$$\Rightarrow \quad \begin{array}{l} \sigma(5) = 5 \\ \sigma(6) = 6 \end{array} \text{ or } \begin{array}{l} 6 \\ 5 \end{array}$$

$$\begin{array}{lllllll} \sigma(1) = 1 & & 2 & & 3 & & 4 \\ \sigma(2) = 2 & & 3 & & 4 & & 1 \\ \sigma(3) = 3 & \text{or} & 4 & \text{or} & 1 & \text{or} & 2 \\ \sigma(4) = 4 & & 1 & & 2 & & 3 \\ \uparrow e & & \uparrow \tau & & (1,3)(2,4) = \tau^2 & & \tau^3 \end{array}$$

①

$\tau^2 = (1,3)(2,4)$ not a a 4-cycle

$\tau^3 = (4,3,2,1)$ is a 4-cycle.

OR
$$\begin{array}{llllllll} \sigma(1) = & 4 & & 3 & & 2 & & 1 \\ \sigma(2) = & 3 & \text{or} & 2 & \text{or} & 1 & \text{or} & 4 \\ \sigma(3) = & 2 & & 1 & & 4 & & 3 \\ \sigma(4) = & 1 & & 4 & & 3 & & 2 \\ & \tau & & (1,4)(2,3) & & (1,3) & & (1,2)(3,4) \quad (2,4) \end{array}$$

$\therefore N_G(H)$ is generated by H (σ, τ)

and $(1,4)(2,3)$, $(1,3)$, $(1,2)(3,4)$, $(2,4)$

$(1,2,3,4) = (1,4)(1,3)(1,2)$.

$(1,4)(2,3)(1,3) = (1,2)(3,4)$

$(1,2,3,4)(13) = (1,4)(2,3)$

$(1,2,3,4)(24) = (1,2)(3,4)$

$$\therefore N_G(H) = H \cup \{(1,4)(2,3), (1,3), (1,2)(3,4), (2,4)\}.$$

Group of order 12.

Example: Suppose G is a group of order 99.

$\therefore \exists$ Sylow subgroups of order 9 & 11.
$ H_3 \qquad H_{11}$

$\Rightarrow H_3$ is abelian, $\cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$

$H_{11} \cong \mathbb{Z}_{11}$

$N_3 = \#$ of Sylow 3-subgroups

$N_3 \mid 11 \to N_3 = 1$ or 11

$N_3 \equiv 1 \bmod 3 \Rightarrow N_3 = 1 \Rightarrow H_3$ is normal!

$N_{11} \mid 9 \to N_{11} = 1, 3,$ or 9

$N_{11} \equiv 1 \bmod 11 \Rightarrow N_{11} = 1 \Rightarrow H_{11}$ is normal!

What's $H_{11} \cap H_3$?

$h \in H_{11} \Rightarrow o(h) = 11$ or $h = e$

$h \in H_3 \Rightarrow o(h) = 3$ or 9 or $h = e$

$\Rightarrow H_{11} \cap H_3 = \{e\}$.

Notice if $h \in H_{11}$, $k \in H_3$

$hk = kh_2$ for some h_2 in H_{11} since H_{11} is normal

$hk = k_2 h$ for some k_2 in H_3 since H_3 is normal

$\Rightarrow kh_2 = k_2 h$

$\Rightarrow \underbrace{k_2^{-1}k}_{\in K} = \underbrace{hh_2^{-1}}_{\in H} \Rightarrow k_2^{-1}k = hh_2^{-1} \in K \cap H = \{e\}$

$\Rightarrow k_2 = k,\ h_2 = h$

$\Rightarrow hk = kh.$

$\Rightarrow G \cong H_3 \times H_{11}$

$\Rightarrow$ $\boxed{\begin{array}{l} G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \\ \text{or } G \cong \mathbb{Z}_9 \times \mathbb{Z}_{11} \end{array}}$